Fermi Problems in Primary Mathematics Classrooms: Pupils' Interactive Modelling Processes

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This paper is based on the results of a 4-year study investigating primary pupils' real-world problem solving and modelling strategies. In order to foster and highlight cooperative mathematical modelling processes, different real-world related Fermi problems were given to three grade 3 and 4 classes. Interpretative analyses of all group work episodes from these classrooms suggest that while most groups did not develop and implement a solution plan, in most cases multiple modelling cycles led to highly appropriate solutions.

Numerous studies in the last two decades have highlighted the difficulties pupils (and teachers) experience when dealing with real-world related word problems (e.g. see Verschaffel, Greer, & deCorte, 2000; Reed, 1999; de Lange, 1998). These difficulties are not only related to complex, non-routine problems but already occur with respect to routine problems that involve the application of a simple algorithm.

Due to difficulties with the comprehension of the text and the identification of the "mathematical core" of the problem, primary school children frequently engage in a rather arbitrary and random operational combination of the numbers given in the text. In doing so, they fail to acknowledge the relationship between the given data and the real-world context. Failure in solving so called "real-world problems" is obviously not related to a lack of practice. In a quantitative study, Renkl and Stern (1994) who analysed the data of 568 pupils from a total of 33 German primary classrooms found that the success rate in solving traditional word problems is not significantly improved by repeated practice.

Real-world problem solving involves the "mathematisation of a non-mathematical situation" (Winter, 1994), that is:

- the construction of a mathematical model with respect to the real-world situation,
- the finding (calculation) of the unknown, and
- and the transfer of the mathematical result derived from the mathematical model to the real-world situation.

Research suggests that the greatest difficulty in this process relates to the identification of an appropriate mathematical model, which requires context knowledge of the real-world situation as well as creativity (Winter, 1994).

However, the last stage of this modelling process, the transfer of the (arithmetical) result to the real-world situation, also presents children with unexpected problems. Verschaffel and De Corte (1997) for example found that fifth graders frequently believe that "37.5 jeeps" is the correct answer to the following problem: *300 soldiers have to be transported by jeep to their training site. Each jeep can hold 8 soldiers. How many jeeps are needed?* This answer (the arithmetically correct result of the division 300 ÷ 8) is furthermore resistant to correction, because children tend to persist with this answer, even when they are questioned about its sense. Checking the result of the division 300 ÷ 8 = 37.5 confirms their understanding that the answer "37.5 jeeps are needed for transportation" is the arithmetically correct result, and hence the right answer. In this context, Freudenthal (1984) points out the construction of a "magical compatibility" with respect to word problems. Because pupils frequently fail to relate the fictive content of the word problem to their real-life experiences, from their point of view the solution of a word problem does not need to match reality (Verschaffel, De Corte, & Lasure, 1999).

The majority of previous publications on word problem solving in primary mathematics either focus on the quantitative analysis of teaching and learning difficulties or the description of problem types, their level of difficulty and/or their potential for the teaching and learning of mathematical modelling (e.g. Silver, 1995). The mathematical modelling process on the other hand has not been given much attention in qualitative research on problem solving in primary mathematics. Furthermore, Pehkonen (1991) pointed out the need for the investigation of problem-solving strategies and modelling processes in primary mathematics *classrooms* in order to complement our scientific knowledge which is currently mainly based on clinical investigations. In addition, Pateman (1996) emphasised the importance of the analysis of peer interaction processes in mathematics learning so that teachers can use this "knowledge to positive effect in creating a classroom in which children learn mathematics" (p. 319).

The 4-year-study which provides the basis of this paper was initiated with the concern that little is known about primary students' real world problem-solving strategies and modelling processes as related to the dynamics of group problem solving in an authentic classroom setting. Although we are reasonably well aware of the individual capabilities of single students, the complexity of the "normal" classroom in terms of mixed abilities, the impact of the variety of socio-cultural experiences of children as well as their quantitative and qualitative participation in classroom interaction have been almost completely neglected. Hence, this classroom-based qualitative study, which was conducted in collaboration with 23 preservice teachers, aimed to investigate the following research questions:

1. Which problem-solving strategies and modelling processes do third and fourth graders explore when solving open real-world problems in small groups?
2. How does the interaction of the group members impact on the solution process?
3. How far can preservice teachers benefit from their involvement in interpretative classroom research in terms of understanding pupils' strategies, modelling processes and interaction patterns?

While the results with respect to the second and third research question have already been presented in previous publications (e.g. Peter-Koop, 2002; Peter-Koop & Wollring, 2001) this paper focuses on the analysis of primary children's modelling processes.

Theoretical Framework: Models of the Modelling Process

Heymann (2003) portrays the concept of mathematical modelling in a recent book on the role of mathematics for general education:

> The concept of modeling can describe the applicability of mathematics and its relation to the "real" world in a very general and, at the same time, quite elementary way. Whenever mathematics is applied to describe and clarify objective situations and to solve real problems, a mathematical model is constructed (or, recourse is taken to an already existing model). Assertions about the relevant situation or solutions to the problem under examination resulting from the use of the model are not valid in isolation from the model. They are in need of interpretation and must be checked for their appropriateness (p. 130).

With respect to the need for interpretation of the solution, the dominance of its arithmetical correctness for primary pupils has been outlined above (see Freudenthal, 1984).

In order to help pupils with respect to solving word problems, German mathematics teachers often promote a systematic 3-step-approach: find the question – do the calculation – give the answer. This scheme is closely related to the four stages of the mathematical problem-solving process described by Pólya (1957, pp. 5-16): (1) *Understanding the problem*, (2) *Devising a plan*, (3) *Carrying out the plan*, and (4) *Looking back*. Based on Pólya's distinction, in German curriculum documents the modelling process is frequently modelled itself in the following rather simplified way (see Figure 1, Müller & Wittmann, 1984, p. 25). The processes represented by the three arrows (*modelling*, *data processing in the model*, *interpretation*) correspond to the stages 2 to 4.

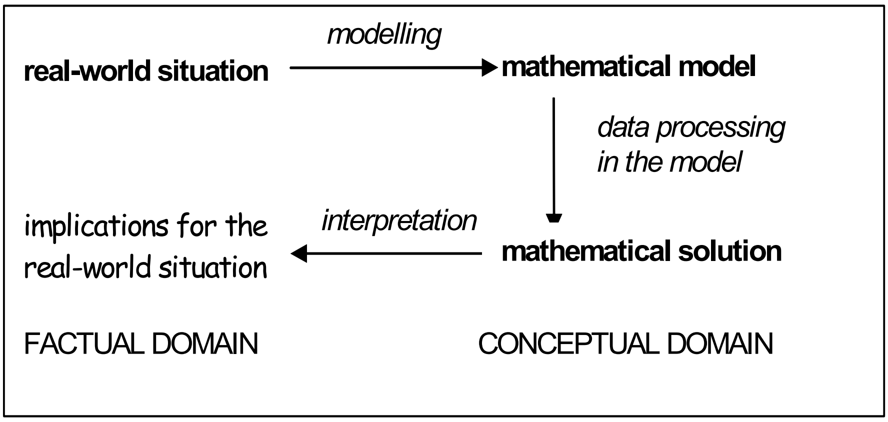

Figure 1. Structure of the modelling process.

In order to understand why many pupils experience difficulties with traditional word problems that often "only" require the application of a taught procedure or algorithm, it may be helpful to investigate on which basis and experiential background primary school children develop mathematical models.

In early childhood mathematics, pupils frequently do not experience mathematics as a tool to solve real-world problems; mathematics itself is visualised by representing sets of various objects (marbles, teddies etc.) by numbers. Hence, the structure of the modelling process (see Figure 1) can also be modelled in the reverse direction as a visualisation/ illustration process (see Figure 2).

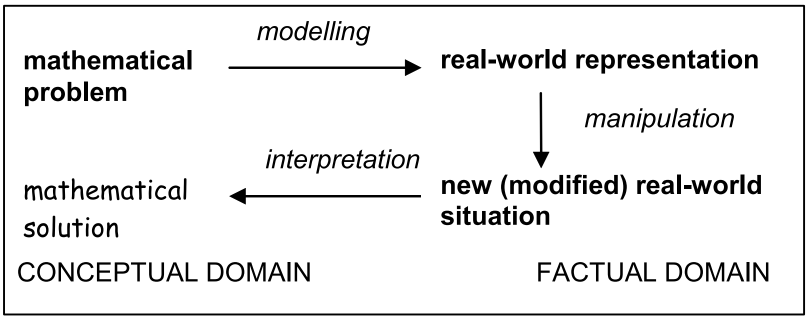

Figure 2. Structure of the visualisation/illustration process.

Simple addition problems such as 3 + 4 for example, are frequently illustrated with a little story: *Anna has 3 marbles and Peter has 4. How many marbles do they have together?* In the classroom situation, Anna's 3 marbles and Peter's 4 marbles are combined and counted. The combining of the two sets with 3 and 4 objects to one set with a total of 7 objects, that is the manipulation of a real-world situation in order to create a new or modified real-world situation, illustrates the addition process. Thus, objects not numbers are represented first. In this context, mathematical modelling can be understood as a reverse procedure to trusted and subconscious processes of visualisation/illustration developed in junior primary mathematics. Traditional word problems hence may not be the best context for developing mathematical modelling skills, because often the particular wording of the problem already suggests the choice of operation to be connected with the given numbers, thus the relationship between the real-world context and mathematics does not need to be explored.

Context of the Study: Fermi Problems

In order to foster and highlight cooperative mathematical modelling processes *Fermi problems* have been used in this classroom based study. Enrico Fermi (1901-1954), who in 1938 won the Nobel Prize for physics for his work on nuclear processes, was known by his students for posing open problems that could only be solved by giving a reasonable estimate. Fermi problems such as *How many piano tuners are there in Chicago?* share the characteristic that the initial response of the problem solver is that the problem could not possibly be solved without recourse to further reference material. However, while individuals frequently reject these problems as too difficult, Clarke and McDonough (1989) pointed out that “pupils, working in cooperative groups, come to see that the knowledge and processes to solve the problem already reside within the group” (p. 22). The following criteria guided the development/choice of problems used in the research project:

- The problems should present challenges and intrinsically motivate cooperation and interaction with peers.
- The wording of the problems should not contain numbers in order to avoid that the children immediately start calculating without first analysing the context of the given situation, and to challenge pupils to engage in estimation and rough calculation and/or the collection of relevant data.
- The problems should be based on a selection of real-world related situations that include *reference contexts* for third and fourth graders.
- The problems should be open-beginning as well as open-ended real-world related tasks that require decision making with respect to the modelling process.

Overall, four such problems have been posed in the following order in one grade 3 and two grade 4 mathematics classes from three different urban schools in a large city in north-western Germany:

1. How much paper does your school use in one month? *(paper problem)*
2. How many children are as heavy as a polar bear? (*polar bear problem*)
3. Your class is planning a trip to visit the Cologne Cathedral. Is it better to travel by bus or by train? (*cathedral problem*)
4. There is a 3 km tailback on the A1 motorway between Muenster and Bremen. How many vehicles are caught in this traffic jam? (*traffic problem*)

The *traffic problem* was selected for the discussion of results in this paper. After the introduction of the problem to the whole class, the class was divided into working groups of four to five children, and each group was videotaped while solving the problem in a separate room in order to ensure sufficient audio quality of the recording. When all groups had finished their work, they presented and explained their solution to the whole class using an overhead transparency of their working. These *strategy conferences* (Peter-Koop, 2003) were also videotaped and analysed.

Methodology: Interpretative Classroom Research

The methodological framework of the project is based on the *interpretative research paradigm* (Bauersfeld, Krummheuer, & Voigt, 1988). Researchers conducting interpretative classroom research seek to investigate typical structures by analysing single cases which are regarded as exemplary. Their focus is the "universal in the special case" and the goal of the interpretation is to comprehensively perceive and understand the (inter)actions of the observed individuals. The significance of the interpretative research paradigm is related to an international change from content-based and individual-psychological approaches towards interpersonal human relations in (mathematics) education.

The data collection and interpretation of the study involved preservice teachers as teacher-researchers (Peter-Koop, 2001) following a strict analytical procedure based on four stages (for details regarding this procedure see Peter-Koop & Wollring, 2001):

1. *video recordings* of all group work episodes (all groups in one class were simultaneously videotaped);
2. *comprehensive transcriptions* of each group work episode with respect to either the full document or selected segments of the recording relevant to the specific research question(s);
3. the *sequential interpretation* of the data by an "interpretation team" of four preservice teachers;
4. the *specific interpretation* of the results on the basis of relevant literature and research findings by an individual preservice teacher-researcher.

While each preservice teacher was in charge of the analysis of one group work episode, the project coordinator and author of this paper was responsible for the meta-analysis, that is the connection and discussion of the findings of these sub-studies.

Data Analysis and Discussion of Selected Results

Due to the complexity of the complete study, the examples chosen in this section to illustrate the results with respect to pupils' modelling processes focus on one group work episode on the *traffic problem* from a grade 4 class. Further examples of student work and their analyses will be discussed during the presentation. The pupils in this grade 4 class were invited to form their own groups which led to a group formation primarily based on friendship, gender and similar ability. Three out of the four groups in this class were gender homogenous (Peter-Koop, 2003). The members of two of these groups (one male, one female) were also all so called "low achievers".

Immediately after the video recordings of the group work episodes, the preservice teachers involved expressed the impression that the work in the groups was "rather chaotic" and "haphazard". However, the groups' working sheets (see Figure 3 for an example) suggest, that most groups – including the low achievers – were highly successful in finding an appropriate solution to the question of how many cars would be caught in the 3 km traffic jam.

While the preservice teachers were clearly impressed with the accuracy of the children's estimates, at the same time most of them were doubtful as to whether the group members would be able to present and explain their solution to the whole class. As it turned out, with the exception of one mixed-ability group, all groups from this grade 4 class managed to explain their solution process (Peter-Koop, 2003).

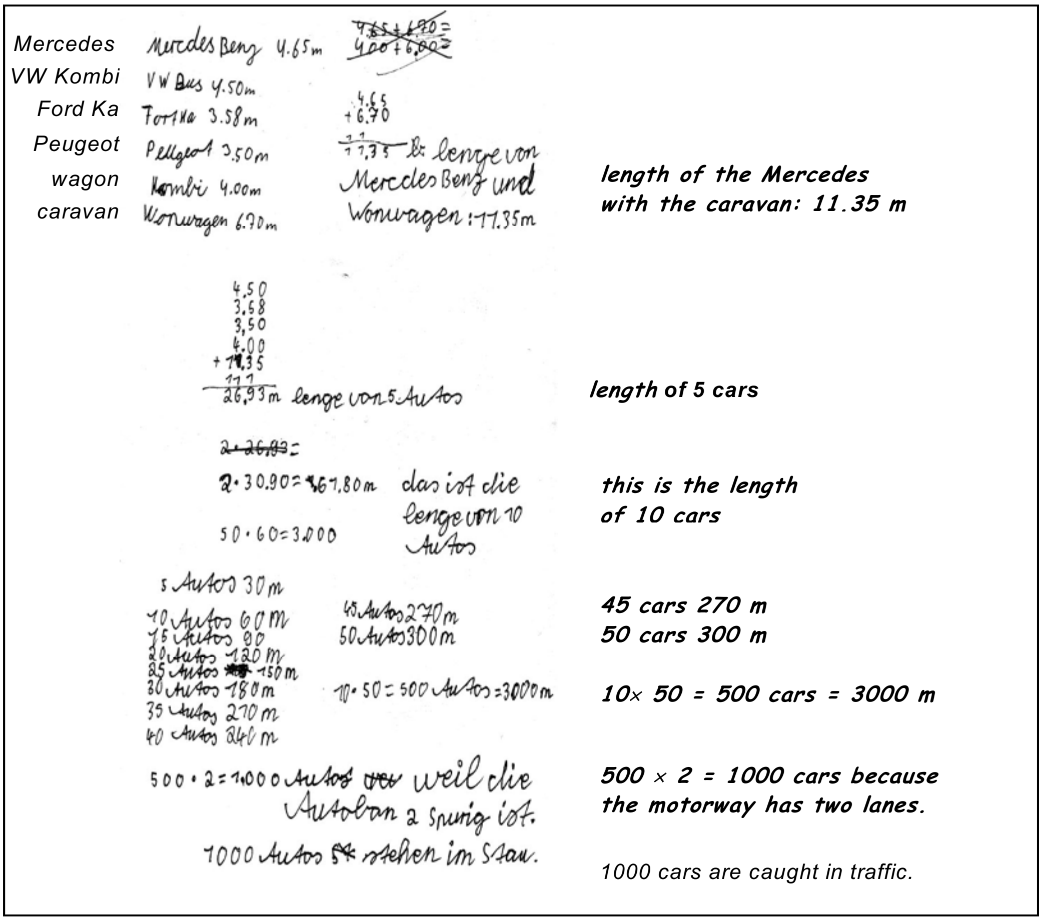

Figure 3. Solution of group C (four low-achieving fourth graders).

The interpretative analyses to a large degree confirm the preservice teachers' initial impressions. In general, the sequential analyses of the transcripts reveal a rather aimless and unstructured approach. Most groups did not develop and then execute a *solution plan* as suggested by Pólya (1957). Hence, the modelling processes did not match the strategy taught in class (question – calculation – answer). However, *understanding the problem* (the first stage described by Pólya) obviously played an important role in the solution, because extensive discussions of the real-word situation and related personal experiences generally preceded the mathematical modelling process. Furthermore, the analyses frequently identified a slowly developing process in which hypotheses were generated, tested, confirmed or neglected while arithmetic results were interpreted leading to the development of further solution ideas. However, the literature suggests only one modelling cycle (see Figure 1). The interpretative analyses in contrast suggested an interweaving of the *factual* and the *conceptual domain*. But the transcripts of the group work turned out to be too detailed to enable identification of the underlying general structure of the pupils' modelling processes. Therefore the transcripts were condensed in the format of *episode plans* in order to highlight the stages characterising the mathematical modelling approach. These episode plans indicate that the different stages outlined in the literature are revisited several times. Figure 4 shows the modelling process as demonstrated by the four boys in group C. While this "multi-cyclic" process is representative of the general structure of the modelling processes identified in this study, the analysis of other group work episodes also identified more complex procedures.

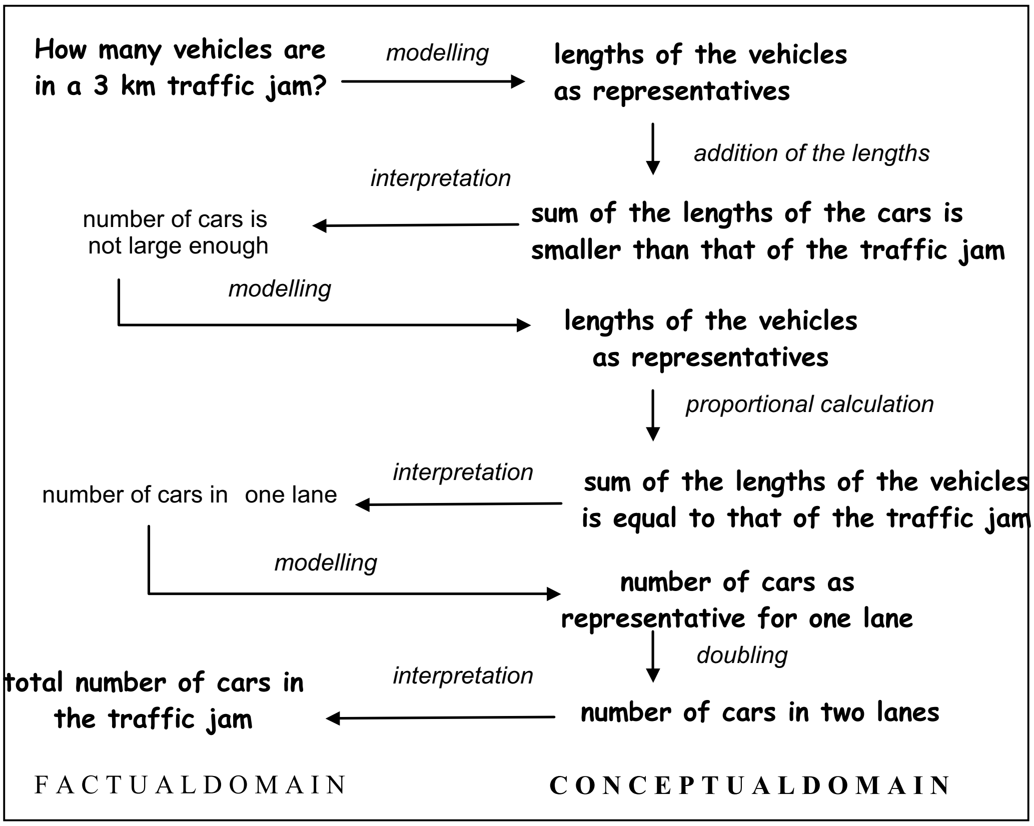

Figure 4. Graphical representation of the modelling process of group C.

Conclusions and Implications

The analysis of the classroom based data suggests that word problems with a high level of complexity (such as Fermi problems) can be solved in sensible and appropriate ways by third and fourth graders. While in traditional problem solving at primary school level only one modelling cycle is needed, Fermi problems can serve as “model-eliciting tasks” (Lesh & Doerr, 2000, p. 380), because the required modelling process extends beyond the application of a standard algorithm and necessitates multiple modelling cycles with multiple ways of thinking about givens, goals, and solution paths (Bell, 1993). Lesh and Doerr (2000) point out that *model development is learning*. In this context, the representative case study from a grade 4 class shown above indicates that the outcome of the modelling activity is a *conceptual tool* that exceeds the solution of a specific problem. During their solution process, the four fourth graders (supposedly low achievers) of group C developed new mathematical knowledge. Their working (see Figure 3) as well as the sequential interpretations of the group work documents that they have discovered the concept of proportionality in using proportional calculation in order to determine the number of cars in a 3 km tailback based on their estimate that 5 cars occupy approximately 30 m.

The results of this study agree with the findings from the analyses of secondary students’ modelling processes by Lesh and Doerr (2000) who highlighted “the need for teachers to examine students’ developing models in order to assess student knowledge and understanding and to foster continued model development in ways that evolve as the student models evolve” (p. 375). Qualitative analyses of the preservice teachers’ learning processes document that they had profoundly underestimated primary pupils’ mathematical modelling competencies (Peter-Koop, 2003). In addition, this study revealed multi-cyclic modelling processes in contrast to a single modelling cycle as suggested in the literature. Hence, it can be concluded that mathematical modelling processes should be given more attention in mathematics teacher education as well as mathematics education research.

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